Hodge Theory and Deformed WZW Models

Based on 2112.00031 with Thomas Grimm

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Plan for the talk

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Hodge Theory

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λ-deformed WZW

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Motivation

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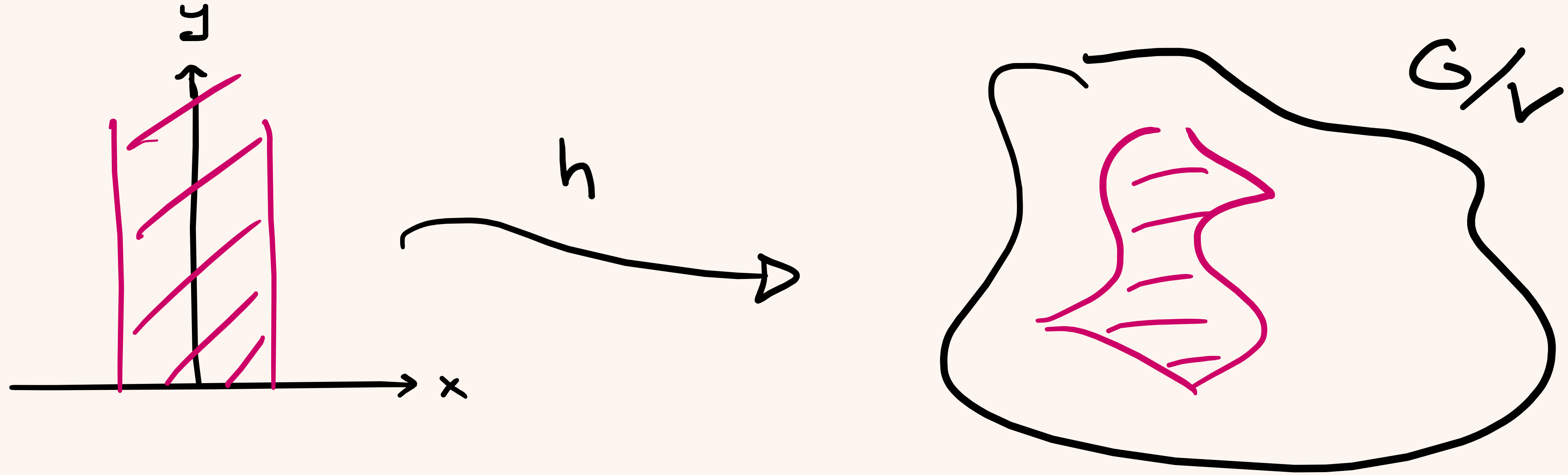
Various frameworks are used:

period vector	**period map**	**Hodge structure**
$\Pi^I = \int_{\gamma_I} \Omega$	$h : \mathcal{M} \to G/V$	$H^3(Y_3, \mathbb{C}) = \bigoplus_{p+q=3} H^{p,q}$

Can think of the period map as a **group-valued field** on the moduli space, with some gauge symmetry group V.



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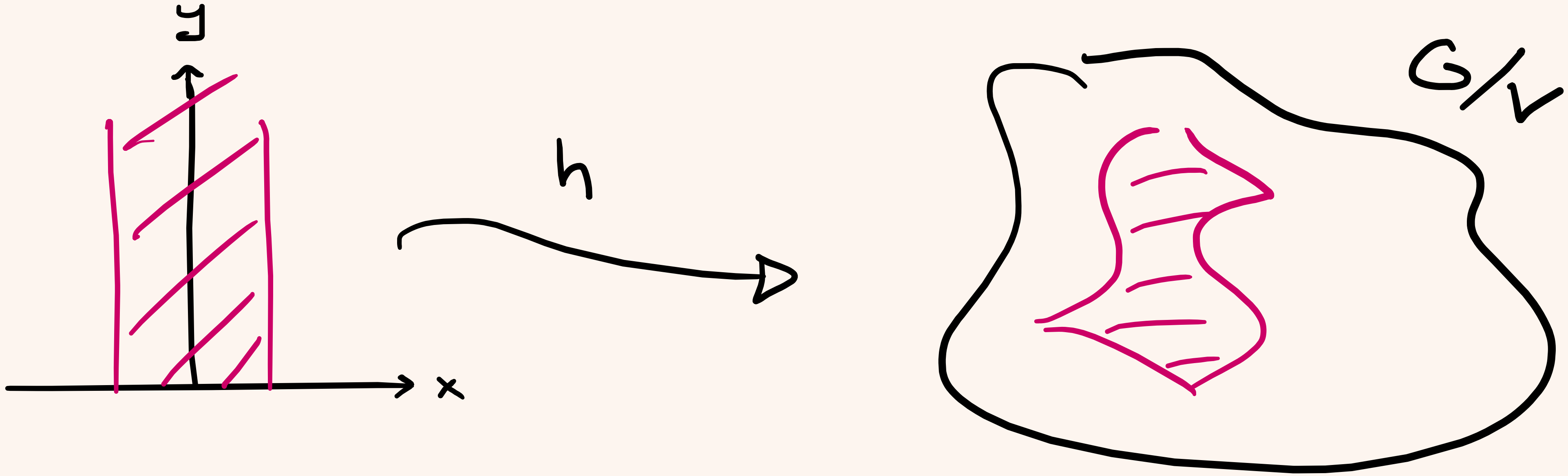


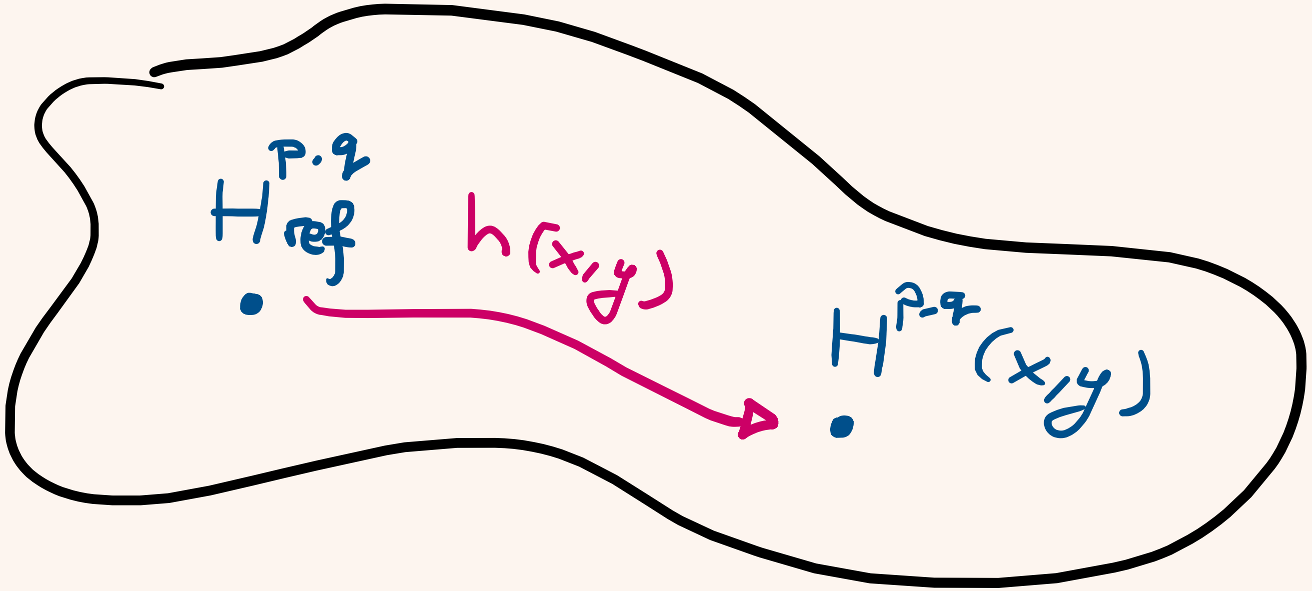
Idea: formulate dynamics in terms of a non-linear sigma model

[Grimm '20] [Cecotti '20]

The Period Map

Introduce a real, group-valued field

$$H^{p,q}(x,y) = h(x,y) H^{p,q}_{\mathrm{ref}}$$



To rephrase the data in the reference Hodge structure, introduce the (reference) **charge operator** [Robles ’16; Kerr, Pearlstein, Robles ‘19]

$$Q_{\mathrm{ref}} v = \frac{1}{2}(p-q)v\,, \qquad v \in H^{p,q}_{\mathrm{ref}}\,, \qquad Q_{\mathrm{ref}} \in i\mathfrak{g}$$

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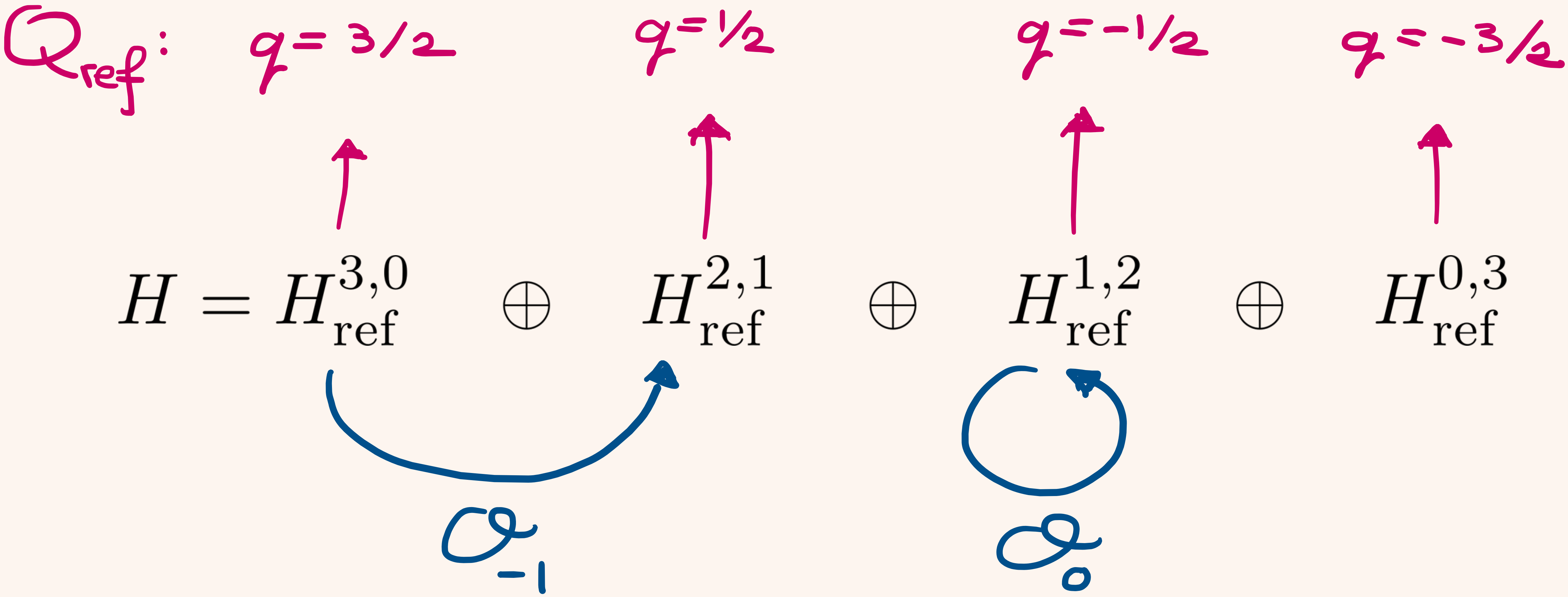
Similarly, get a **charge decomposition** of operators

$$\mathcal{O} = \sum_q \mathcal{O}_q\,, \qquad [Q_{\text{ref}}, \mathcal{O}_q] = q\,\mathcal{O}_q\,, \qquad \mathcal{O} \in \mathfrak{g}_{\mathbb{C}}$$



$$H = H_{\text{ref}}^{3,0} \oplus H_{\text{ref}}^{2,1} \oplus H_{\text{ref}}^{1,2} \oplus H_{\text{ref}}^{0,3}$$

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The period map satisfies the **horizontality condition** [Griffiths '68; Schmid '73]

$$h^{-1}\partial_{\pm}h = (h^{-1}\partial_{\pm}h)_0 + (h^{-1}\partial_{\pm}h)_{\pm 1}$$

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A more familiar relation is perhaps

$$\Omega \in H^{3,0}\,, \qquad \partial\Omega \in H^{3,0} \oplus H^{2,1}\,, \qquad \cdots$$

λ-Deformed WZW Models

Action of the λ-deformed WZW model: [Sfetsos '14]

$$S_{k,\lambda}[g] = S_{\mathrm{WZW},k}[g] + \lambda \frac{k}{\pi} \int d^2\sigma \, \mathrm{Tr}\left(J_+ \left(1 - \lambda \, \mathrm{Ad}_g\right)^{-1} J_- \right)$$

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Can be viewed as an all-loop effective action of the non-Abelian bosonized Thirring model [Itsios, Sfetsos, Siampos '14]

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has many interesting properties:

1. Integrable
2. \`S-duality' $S_{-k,\lambda^{-1}}[g^{-1}] = S_{k,\lambda}[g]$
3. Interpolates between WZW (λ=0) and non-Abelian T-dual of PCM (λ=1)
4. Poisson-Lie T-dual to η-model (|λ|=1)

Gauged version: [Hollowood, Miramontes, Schmidtt '14]

$$S_{G/G,\lambda}[g,A] = S_{\mathrm{WZW},k}[g] - \frac{k}{\pi}\int d^2\sigma\, \mathrm{Tr}\left(A_- J_+ + A_+ J_- + A_- g A_+ g^{-1} - \lambda^{-1} A_+ A_-\right)$$

Gauge field acts as a Lagrange multiplier

$$A_\pm = \frac{\lambda}{1 - \lambda\, \mathrm{Ad}_{g^{\pm 1}}} J_\pm$$

On-shell, recover the λ-deformed WZW model.

Hodge Theory from λ-deformations

[Grimm, JM ‘21]

A natural ansatz is given by

$$g = z^{Q}\,, \qquad A_{\pm} = \pm i\alpha\,\partial_{\pm} Q\,, \qquad Q = hQ_{\mathrm{ref}}h^{-1}$$

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E.o.m. of the gauge fields precisely constrains the charge components

$$(h^{-1}\partial_{\pm}h)_{\pm q} = 0\,, \qquad \text{unless} \qquad b(q) := \alpha q - \frac{1 - z^{q}}{\lambda^{-1} - z^{q}} = 0$$

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special case: z = -1 (Weil operator)

$$h^{-1}\partial_{\pm}h = (h^{-1}\partial_{\pm}h)_{0} + (h^{-1}\partial_{\pm}h)_{\pm 1}$$

The remaining e.o.m. of g furthermore impose

$$\lambda = \pm i \qquad + \qquad \text{Nahm's equations}$$

The latter can be interpreted as arising from minimizing the worldvolume of the string w.r.t. the **Hodge metric**.

Conclusion

One-parameter period maps provide a subset of the classical solutions to the λ-deformed WZW model.

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Next up:

1. Insights into asymptotic Hodge theory?
2. Role of integrability?

Thank you!